

# A note on global regularity results for 2D Boussinesq equations with fractional dissipation

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**Abstract:** In this paper we study the Cauchy problem for the two-dimensional (2D) incompressible Boussinesq equations with fractional Laplacian dissipation and thermal diffusion. Invoking the energy method and several commutator estimates, we get the global regularity result of the 2D Boussinesq equations as long as  $1 - \alpha < \beta < \min \left\{ \frac{\alpha}{2}, \frac{3\alpha-2}{2\alpha^2-6\alpha+5}, \frac{2-2\alpha}{4\alpha-3} \right\}$  with  $0.77963 \approx \alpha_0 < \alpha < 1$ . As a result, this result is a further improvement of the previous two works [32, 42].

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## 1. INTRODUCTION

In this paper we study the Cauchy problem for the 2D incompressible Boussinesq equations with fractional Laplacian dissipation in  $\mathbb{R}^2$

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nu \Lambda^\alpha u + \nabla p = \theta e_2, \\ \partial_t \theta + (u \cdot \nabla)\theta + \kappa \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \quad (1.1)$$

where  $u(x, t) = (u_1(x, t), u_2(x, t))$  is a vector field denoting the velocity,  $\theta = \theta(x, t)$  is a scalar function denoting the temperature in the content of thermal convection and the density in the modeling of geophysical fluids,  $p$  the scalar pressure and  $e_2 = (0, 1)$ . Here the numbers  $\nu \geq 0$ ,  $\kappa \geq 0$ ,  $\alpha \geq 0$  and  $\beta \geq 0$  are real parameters. The fractional Laplacian operator  $\Lambda^\alpha$ ,  $\Lambda := (-\Delta)^{\frac{1}{2}}$  denotes the Zygmund operator which is defined through the Fourier transform, namely

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx.$$

The fractional Laplacian serves to model many physical phenomena such as overdriven detonations in gases [10]. It is also used in some mathematical models in hydrodynamics, molecular biology and finance mathematics, see for instance [16].

Actually, the standard 2D Boussinesq equations (that is  $\alpha = \beta = 2$ ) model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Rayleigh-Bernard convection (see for example [31, 33] and references therein). Moreover, there are some geophysical circumstances related to the Boussinesq equations with fractional Laplacian (see [7, 33] for details). The Boussinesq equations with fractional Laplacian also closely related equations such as the surface quasi-geostrophic equation model important geophysical phenomena (see, e.g., [11]). The standard 2D Boussinesq equations and their fractional Laplacian generalizations have attracted considerable attention recently due to their physical applications and mathematical significance. Obviously, for case  $\mu = \kappa = 0$ , the system (1.1) reduces to the inviscid Boussinesq equations, whose global well-posedness of smooth solutions is an outstanding open problem in fluid dynamics (except if  $\theta_0$  is a constant, of course) which may be formally compared to the similar problem for the three-dimensional axisymmetric Euler equations with swirl (see [31]). In contrast, in the case when  $\alpha = \beta = 2$ , the global well posedness has been shown previously, we refer, for example, to [5]. Therefore, there are a large number of works devoted to studying the intermediate cases, such as fractional dissipation, partial anisotropic dissipation and so on. The global regularity to the system (1.1) for the cases when  $\alpha = 2$  and  $\kappa = 0$  or  $\beta = 2$  and  $\mu = 0$  were established by Chae [8] and by Hou and Li [23] independently. By deeply developing the structures of the coupling system, Hmidi, Keraani and Rousset [20, 21] were able to established the global well-posedness result to the system (1.1) with two special critical case, namely  $\alpha = 1$  and  $\kappa = 0$  or  $\beta = 1$  and  $\mu = 0$ . The more general critical case  $\alpha + \beta = 1$  with  $0 < \alpha, \beta < 1$  is extremely difficult. Very recently, the global regularity of the general critical case  $\alpha + \beta = 1$  with  $\alpha > \frac{23-\sqrt{145}}{12} \approx 0.9132$  and  $0 < \beta < 1$  was recently examined by Jiu, Miao, Wu and Zhang [25]. This result was further improved by Stefanov and Wu [34], which requires  $\alpha + \beta = 1$  with  $\alpha > \frac{\sqrt{1777}-23}{24} \approx 0.798$  and  $0 < \beta < 1$ . Here we want to state that even in the subcritical ranges, namely  $\alpha + \beta > 1$  with  $0 < \alpha, \beta < 1$ , the global regularity of (1.1) is also definitely nontrivial and quite difficult. Actually, to the best of our knowledge there are only several works concerning the subcritical cases, please refer to [12, 32, 38, 40, 41, 42]. More precisely, Miao and Xue [32] obtained the global regularity for system (1.1) for the case  $\nu > 0, \kappa > 0$  and

$$\frac{6 - \sqrt{6}}{4} < \alpha < 1, \quad 1 - \alpha < \beta < \min \left\{ \frac{7 + 2\sqrt{6}}{5} \alpha - 2, \frac{\alpha(1 - \alpha)}{\sqrt{6} - 2\alpha}, 2 - 2\alpha \right\}.$$

In addition, Constantin and Vicol [12] verified the global regularity of the system (1.1) on the case when the thermal diffusion dominates, namely

$$\nu > 0, \quad \kappa > 0, \quad 0 < \alpha < 2, \quad 0 < \beta < 2, \quad \beta > \frac{2}{2 + \alpha}.$$

Recently, Yang, Jiu and Wu [38] proved the global regularity of the system (1.1) with

$$\nu > 0, \quad \kappa > 0, \quad 0 < \alpha < 1, \quad 0 < \beta < 1, \quad \beta > 1 - \frac{\alpha}{2}, \quad \beta \geq \frac{2 + \alpha}{3}, \quad \beta > \frac{10 - 5\alpha}{10 - 4\alpha}.$$

Here we want to point out that the above two works [12, 38] have been improved by the recent two manuscripts [40, 41]. In particular, we [41] proved the global well-posedness

result for the system (1.1) with

$$\nu > 0, \quad \kappa > 0, \quad 0 < \alpha < 1, \quad 0 < \beta < 1, \quad \beta > 1 - \frac{\alpha}{2}.$$

It is also worthwhile to mention that there are numerous studies about the Boussinesq equations with partial anisotropic dissipation, see for example [2, 3, 1, 15, 6, 28]. Many other interesting recent results on the Boussinesq equations can be found, with no intention to be complete (see, e.g., [4, 9, 13, 14, 15, 19, 22, 24, 25, 26, 27, 29, 28, 30, 35, 36, 37, 39] and the references therein).

To complement and improve the existing results described above, this paper continues the previous two works [32, 42] to show the global regularity result. Since the concrete values of the constant  $\nu, \kappa$  play no role in our discussion, we shall assume  $\nu = \kappa = 1$  throughout this paper. Now our main result is the following theorem.

**Theorem 1.1.** *Suppose that  $0.77963 \approx \alpha_0 < \alpha < 1$  and  $0 < \beta < 1$  obeys*

$$1 - \alpha < \beta < \min \left\{ \frac{\alpha}{2}, \frac{3\alpha - 2}{2\alpha^2 - 6\alpha + 5}, \frac{2 - 2\alpha}{4\alpha - 3} \right\}. \quad (1.2)$$

*Let  $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2) \times H^\sigma(\mathbb{R}^2)$  for  $\sigma > 2$ , then the system (1.1) admits a unique global solution such that for any  $T > 0$*

$$\begin{aligned} u &\in C([0, T]; H^\sigma(\mathbb{R}^2)) \cap L^2([0, T]; H^{\sigma+\frac{\alpha}{2}}(\mathbb{R}^2)), \\ \theta &\in C([0, T]; H^\sigma(\mathbb{R}^2)) \cap L^2([0, T]; H^{\sigma+\frac{\beta}{2}}(\mathbb{R}^2)). \end{aligned}$$

**Remark 1.2.** Here we say some words about  $\alpha_0$  which can be explicitly formulated as

$$\alpha_0 = \frac{8 - (\sqrt[3]{6\sqrt{609} + 118} - \sqrt[3]{6\sqrt{609} - 118})}{6} \approx 0.77963.$$

By using the well-known Shengjin's Formulas [17], it is easy to show that  $\alpha_0$  is a unique real solution to the following cubic equation

$$2\alpha^3 - 8\alpha^2 + 14\alpha - 7 = 0.$$

**Remark 1.3.** The condition  $\alpha > \alpha_0 \approx 0.77963$  is weaker than the previous two works [32, 42], where the corresponding conditions are  $\alpha > \frac{6-\sqrt{6}}{4} \approx 0.887627$  and  $\alpha > \frac{21-\sqrt{217}}{8} \approx 0.783635$ , respectively. Hence, this result can be regarded as a further improvement of the results in [32, 42].

**Remark 1.4.** For technical reasons, the  $\beta$  should be smaller than a complicated explicit function. As a matter of fact, it is strongly believed that the diffusion term is always good term and the larger the power  $\beta$  is, the better effects it produces. Therefore, we conjecture that the above theorem should hold for all the cases  $\alpha_0 < \alpha < 1$  and  $1 - \alpha < \beta < 1$ .

## 2. THE PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. Now let us to prove our main theorem. First, the local well posedness of the system (1.1) for smooth initial data is well-known to us (see for example [31]), and therefore, it suffices to prove the global in time *a priori* estimate on  $[0, T]$  for any given  $T > 0$ . In this paper, all constants will be

denoted by  $C$  that is a generic constant depending only on the quantities specified in the context.

Thanks to the basic energy estimates, we obtain immediately

$$\sup_{0 \leq t \leq T} \|\theta(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac{\beta}{2}} \theta(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2, \quad \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad \forall p \in [2, \infty], \quad (2.1)$$

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac{\alpha}{2}} u(\tau)\|_{L^2}^2 d\tau \leq C(T, u_0, \theta_0). \quad (2.2)$$

Now we apply operator curl to the equation (1.1)<sub>1</sub> to obtain the following vorticity  $w = \partial_1 u_2 - \partial_2 u_1$  equation

$$\partial_t w + (u \cdot \nabla) w + \Lambda^\alpha w = \partial_x \theta. \quad (2.3)$$

However, the "vortex stretching" term  $\partial_x \theta$  appears to prevent us from proving any global bound for  $w$ . To overcome this difficulty, a natural idea is to eliminate the term  $\partial_x \theta$  from the vorticity equation. This method was first introduced by Hmidi, Keraani and Rousset [20, 21] to treat the Boussinesq equations with critical cases. Now we set  $\mathcal{R}_\alpha$  as the singular integral operator

$$\mathcal{R}_\alpha := \partial_x \Lambda^{-\alpha}.$$

Then we can show that the new quantity  $G = \omega - \mathcal{R}_\alpha \theta$  satisfies

$$\partial_t G + (u \cdot \nabla) G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla] \theta + \Lambda^{\beta-\alpha} \partial_x \theta, \quad (2.4)$$

here and in sequel, the following standard commutator notation are used frequently

$$[\mathcal{R}_\alpha, u \cdot \nabla] \theta := \mathcal{R}_\alpha (u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.$$

The above equation is very important in our analysis in order to derive some crucial *a priori* estimates. Moreover, the velocity field  $u$  can be decomposed into the following two parts

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha := u_G + u_\theta.$$

Before further proving our main result, we need to recall some useful lemmas. The first lemma concerns the following commutator estimate, which plays a key role in proving our main result.

**Lemma 2.1** (see [41]). *Let  $p \in [2, \infty)$  and  $r \in [1, \infty]$  and  $\delta \in (0, 1)$ ,  $s \in (0, 1)$  such that  $s + \delta < 1$ , then it holds*

$$\|[\Lambda^\delta, f]g\|_{B_{p,r}^s} \leq C(p, r, \delta, s) (\|\nabla f\|_{L^p} \|g\|_{B_{\infty,r}^{s+\delta-1}} + \|f\|_{L^2} \|g\|_{L^2}). \quad (2.5)$$

Here and in what follows,  $B_{p,r}^s$  denotes the standard Besov space.

To prove the theorem, we need the following commutator estimate involving  $\mathcal{R}_\alpha$ , which was established by Stefanov and Wu [34].

**Lemma 2.2.** *Assume that  $\frac{1}{2} < \alpha < 1$  and  $1 < p_2 < \infty$ ,  $1 < p_1, p_3 \leq \infty$  with  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ . Then for  $0 \leq s_1 < 1 - \alpha$  and  $s_1 + s_2 > 1 - \alpha$ , the following holds true*

$$\left| \int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u_G \cdot \nabla] \theta dx \right| \leq C \|\Lambda^{s_1} \theta\|_{L^{p_1}} \|F\|_{W^{s_2, p_2}} \|G\|_{L^{p_3}}. \quad (2.6)$$

Similarly, for  $0 \leq s_1 < 1 - \alpha$  and  $s_1 + s_2 > 2 - 2\alpha$ , the following holds true

$$\left| \int_{\mathbb{R}^2} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla] H \, dx \right| \leq C \|\Lambda^{s_1} \theta\|_{L^{p_1}} \|F\|_{W^{s_2, p_2}} \|H\|_{L^{p_3}}. \quad (2.7)$$

Here and in what follows,  $W^{s,p}$  denotes the standard Sobolev space.

The following lemma is the bilinear estimate which will be used frequently.

**Lemma 2.3.** *Let  $2 < m < \infty$ ,  $0 < s < 1$  and  $p, q, r \in (1, \infty)^3$  such that  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ , then it holds*

$$\|\Lambda^s(|f|^{m-2}f)\|_{L^p} \leq C \|f\|_{\dot{B}_{q,p}^s} \|f\|_{L^{r(m-2)}}^{m-2}, \quad (2.8)$$

$$\| |f|^{m-2}f \|_{W^{s,p}} \leq C \|f\|_{B_{q,p}^s} \|f\|_{L^{r(m-2)}}^{m-2}.$$

**Proof of Lemma 2.3.** One can find the proof in [42] and we sketch it here for convenience. Let us recall the following characterization of  $\dot{W}^{s,p}$  with  $0 < s < 1$

$$\|\Lambda^s(|f|^{m-2}f)\|_{L^p}^p \approx \int_{\mathbb{R}^2} \frac{\| |f|^{m-2}f(x + \cdot) - |f|^{m-2}f(\cdot) \|_{L^p}^p}{|x|^{2+sp}} \, dx.$$

Note that the following simple inequality

$$\left| |a|^{m-2}a - |b|^{m-2}b \right| \leq C(m) |a - b| (|a|^{m-2} + |b|^{m-2}),$$

and Hölder inequality, it results in

$$\begin{aligned} \| |f|^{m-2}f(x + \cdot) - |f|^{m-2}f(\cdot) \|_{L^p} &\leq C \|f(x + \cdot) - f(\cdot)\|_{L^q} \| |f|^{m-2}f \|_{L^r} \\ &\leq C \|f(x + \cdot) - f(\cdot)\|_{L^q} \|f\|_{L^{r(m-2)}}^{m-2}. \end{aligned}$$

Thus, it follows from the characterization of Besov space that

$$\begin{aligned} \|\Lambda^s(|f|^{m-2}f)\|_{L^p}^p &\leq C \int_{\mathbb{R}^2} \frac{\|f(x + \cdot) - f(\cdot)\|_{L^q}^p \|f\|_{L^{r(m-2)}}^{(m-2)p}}{|x|^{2+sp}} \, dx \\ &\leq C \|f\|_{L^{r(m-2)}}^{(m-2)p} \int_{\mathbb{R}^2} \frac{\|f(x + \cdot) - f(\cdot)\|_{L^q}^p}{|x|^{2+sp}} \, dx \\ &\leq C \|f\|_{L^{r(m-2)}}^{(m-2)p} \|f\|_{\dot{B}_{q,p}^s}^p. \end{aligned}$$

The Hölder inequality directly gives

$$\| |f|^{m-2}f \|_{L^p} \leq C \|f\|_{L^q} \| |f|^{m-2}f \|_{L^r} = C \|f\|_{L^q} \|f\|_{L^{r(m-2)}}^{m-2}.$$

Consequently, this concludes the proof of the lemma.  $\square$

With the above lemmas in hand, we continue to prove the main result. First we are now in the position to derive the following estimate concerning the temperature  $\theta$  and  $G$ , which plays an important role in proving the main theorem and is also the main difference compared to the recent manuscript [42].

**Lemma 2.4.** *Under the assumptions stated in Theorem 1.1, let  $(u, \theta)$  be the corresponding solution of the system (1.1). If  $\beta > 1 - \alpha$  and  $\alpha > \frac{2}{3}$ , then the temperature  $\theta$  admits the following bound for any  $\max \left\{ \frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2} \right\} < \delta < \frac{\beta}{2}$*

$$\sup_{0 \leq t \leq T} (\|G(t)\|_{L^2}^2 + \|\Lambda^\delta \theta(t)\|_{L^2}^2) + \int_0^T (\|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2)(\tau) \, d\tau \leq C(T, u_0, \theta_0),$$

(2.9)

where  $C(T, u_0, \theta_0)$  is a constant depending on  $T$  and the initial data.

**Remark 2.5.** Although the above estimate (2.9) holds for  $\max\left\{\frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2}\right\} < \delta < \frac{\beta}{2}$ , yet by energy estimate (2.1) and the classical interpolation, we find that (2.9) is actually true for any  $0 \leq \delta < \frac{\beta}{2}$ .

**Proof of Lemma 2.4.** Applying  $\Lambda^\delta$  ( $\delta > 0$  to be fixed later) to (1.1)<sub>2</sub>, then multiplying it by  $\Lambda^\delta \theta$ , after integration by parts, we find that

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^\delta \theta(t)\|_{L^2}^2 + \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 = - \int_{\mathbb{R}^2} \Lambda^\delta (u \cdot \nabla \theta) \Lambda^\delta \theta \, dx. \quad (2.10)$$

Hence, an application of the divergence-free condition, commutator estimate (2.5), Besov embedding and Gagliardo-Nirenberg inequality directly yields

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \Lambda^\delta (u \cdot \nabla \theta) \Lambda^\delta \theta \, dx \right| \\ &= \left| \int_{\mathbb{R}^2} [\Lambda^\delta, u \cdot \nabla] \theta \, \Lambda^\delta \theta \, dx \right| \\ &= \left| \int_{\mathbb{R}^2} \nabla \cdot [\Lambda^\delta, u] \theta \, \Lambda^\delta \theta \, dx \right| \\ &\leq C \|\Lambda^{1-\frac{\beta}{2}} [\Lambda^\delta, u] \theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C \|[\Lambda^\delta, u] \theta\|_{H^{1-\frac{\beta}{2}}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C \|[\Lambda^\delta, u] \theta\|_{B_{2,2}^{1-\frac{\beta}{2}}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C (\|\nabla u\|_{L^2} \|\theta\|_{B_{\infty,2}^{\delta-\frac{\beta}{2}}} + \|u\|_{L^2} \|\theta\|_{L^2}) \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \quad \left(\delta < \frac{\beta}{2}\right) \\ &\leq C \|\omega\|_{L^2} \|\theta\|_{L^\infty} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} + C \|u\|_{L^2} \|\theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C (\|G\|_{L^2} + \|\mathcal{R}_\alpha \theta\|_{L^2}) \|\theta\|_{L^\infty} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} + C \|u\|_{L^2} \|\theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C \|\theta\|_{L^\infty} \|G\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} + C \|\theta\|_{L^\infty} \|\Lambda^{1-\alpha} \theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\quad + C \|u\|_{L^2} \|\theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\leq C \|\theta\|_{L^\infty} \|G\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} + C \|\theta\|_{L^\infty} \|\theta\|_{L^2}^{\frac{2\delta+\beta+2\alpha-2}{2\delta+\beta}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^{\frac{2-2\alpha}{2\delta+\beta}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\quad + C \|u\|_{L^2} \|\theta\|_{L^2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \\ &\quad \left(\delta > \frac{2-2\alpha-\beta}{2} \Rightarrow \frac{2-2\alpha}{2\delta+\beta} < 1\right) \\ &\leq \frac{1}{2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 + C (\|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}} + \|u\|_{L^2}^2) \|\theta\|_{L^2}^2 + C \|\theta\|_{L^\infty}^2 \|G\|_{L^2}^2. \end{aligned}$$

Here we have applied the following facts

$$L^\infty \hookrightarrow B_{\infty,2}^{\delta-\frac{\beta}{2}}, \quad \|\Lambda^{1-\alpha} \theta\|_{L^2(\mathbb{R}^2)} \leq C \|\theta\|_{L^2(\mathbb{R}^2)}^{\frac{2\delta+\beta+2\alpha-2}{2\delta+\beta}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2(\mathbb{R}^2)}^{\frac{2-2\alpha}{2\delta+\beta}},$$

which holds true for  $\delta < \frac{\beta}{2}$  and  $\delta > \frac{2-2\alpha-\beta}{2}$ , respectively. Substituting the above estimate into (2.10), we arrive at

$$\frac{d}{dt} \|\Lambda^\delta \theta(t)\|_{L^2}^2 + \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 \leq C(\|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}} + \|u\|_{L^2}^2) \|\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2 \|G\|_{L^2}^2, \quad (2.11)$$

Now we test the equation (2.4) by  $G$ , integrate the resulting inequality with respect to  $x$  and make use of divergence-free condition to obtain

$$\frac{1}{2} \frac{d}{dt} \|G(t)\|_{L^2}^2 + \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 = \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u \cdot \nabla] \theta G dx + \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \partial_x \theta G dx. \quad (2.12)$$

We easily deduce from Gagliardo-Nirenberg inequality and Young inequality that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \partial_x \theta G dx \right| &\leq C \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \|\Lambda^{1+\frac{\beta}{2}-\alpha-\delta} G\|_{L^2} \\ &\leq C \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2} \|G\|_{L^2}^{\frac{3\alpha+2\delta-2-\beta}{\alpha}} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^{\frac{2+\beta-2\alpha-2\delta}{\alpha}} \\ &\quad \left( \frac{2+\beta-3\alpha}{2} < \delta < \frac{2+\beta-2\alpha}{2} \right) \\ &\leq \frac{1}{4} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \frac{1}{4} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 + C \|G\|_{L^2}^2, \end{aligned} \quad (2.13)$$

where in the second line, we have used the following Gagliardo-Nirenberg inequality

$$\|\Lambda^{1+\frac{\beta}{2}-\alpha-\delta} G\|_{L^2} \leq C \|G\|_{L^2}^{\frac{3\alpha+2\delta-2-\beta}{\alpha}} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^{\frac{2+\beta-2\alpha-2\delta}{\alpha}},$$

for any  $\frac{2+\beta-3\alpha}{2} < \delta < \frac{2+\beta-2\alpha}{2}$ .

Observing the decomposition  $u = u_G + u_\theta$ , we get

$$\int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u \cdot \nabla] \theta G dx = \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta G dx + \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta G dx.$$

Let us use the estimate (2.6) with  $s_1 = 0$  to control the above first term as

$$\begin{aligned} \left| \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta G dx \right| &\leq C \|\theta\|_{L^\infty} \|G\|_{L^2} \|G\|_{H^{s_2}} \quad (s_2 > 1 - \alpha) \\ &\leq C \|\theta\|_{L^\infty} \|G\|_{L^2} \|G\|_{H^{\frac{\alpha}{2}}} \quad (s_2 \leq \frac{\alpha}{2}) \\ &\leq \frac{1}{8} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + C(1 + \|\theta\|_{L^\infty}^2) \|G\|_{L^2}^2. \end{aligned} \quad (2.14)$$

To estimate the second term, we can apply the estimate (2.7) with  $s_2 = \frac{\alpha}{2}$  to conclude that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta G dx \right| &\leq C \|\theta\|_{L^\infty} \|\theta\|_{H^{s_1}} \|G\|_{H^{\frac{\alpha}{2}}} \quad \left( \frac{4-5\alpha}{2} < s_1 < 1 - \alpha \right) \\ &\leq C \|\theta\|_{L^\infty} \|\theta\|_{L^2}^{\frac{2\delta+\beta-2s_1}{2\delta+\beta}} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^{\frac{2s_1}{2\delta+\beta}} \|G\|_{H^{\frac{\alpha}{2}}} \\ &\quad \left( 0 < s_1 < \delta + \frac{\beta}{2} \right) \\ &\leq \frac{1}{8} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \frac{1}{4} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 + C \|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}} \|\theta\|_{L^2}^2 \end{aligned} \quad (2.15)$$

where in the first line and second line, the number  $s_1$  should be satisfied

$$\max \left\{ 0, \frac{4-5\alpha}{2} \right\} < s_1 < \min \left\{ 1-\alpha, \delta + \frac{\beta}{2} \right\},$$

which can be ensured by choosing  $\delta > \frac{4-5\alpha-\beta}{2}$  and  $\alpha > \frac{2}{3}$ .

Inserting above estimates (2.13)-(2.15) into (2.12), we can conclude

$$\frac{1}{2} \frac{d}{dt} \|G(t)\|_{L^2}^2 + \frac{1}{2} \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 \leq \frac{1}{2} \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 + C(1 + \|\theta\|_{L^\infty}^2) \|G\|_{L^2}^2 + C \|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}} \|\theta\|_{L^2}^2. \quad (2.16)$$

By putting (2.11) and (2.16) together, we finally get

$$\begin{aligned} & \frac{d}{dt} (\|G(t)\|_{L^2}^2 + \|\Lambda^\delta \theta(t)\|_{L^2}^2) + \|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2 \\ & \leq C(1 + \|\theta\|_{L^\infty}^{\frac{2(2\delta+\beta)}{2\delta+\beta+2\alpha-2}} + \|\theta\|_{L^\infty}^{\frac{4\delta+2\beta}{2\delta+\beta-2s_1}} + \|u\|_{L^2}^2) \|\theta\|_{L^2}^2 + C(1 + \|\theta\|_{L^\infty}^2) \|G\|_{L^2}^2, \end{aligned} \quad (2.17)$$

for any  $\delta$  satisfying

$$\max \left\{ \frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2}, \frac{4-5\alpha-\beta}{2} \right\} < \delta < \min \left\{ \frac{\beta}{2}, \frac{\beta+2-2\alpha}{2} \right\} = \frac{\beta}{2}.$$

Observing the facts  $\alpha > \frac{2}{3} \Rightarrow \frac{4-5\alpha-\beta}{2} < \frac{2-2\alpha-\beta}{2}$  and  $\frac{\beta}{2} < \frac{\beta+2-2\alpha}{2}$ , the range of  $\delta$  becomes

$$\max \left\{ \frac{2-2\alpha-\beta}{2}, \frac{2+\beta-3\alpha}{2} \right\} < \delta < \frac{\beta}{2}.$$

By the standard Gronwall inequality, we can easily get from (2.17) that

$$\sup_{0 \leq t \leq T} (\|G(t)\|_{L^2}^2 + \|\Lambda^\delta \theta(t)\|_{L^2}^2) + \int_0^T (\|\Lambda^{\frac{\alpha}{2}} G\|_{L^2}^2 + \|\Lambda^{\delta+\frac{\beta}{2}} \theta\|_{L^2}^2)(\tau) d\tau \leq C(T, u_0, \theta_0).$$

Thus the conclusion is proved.  $\square$

Next we establish the following global *a priori* bound of  $L^m$  norm for  $G$  based on Lemma 2.4. This *a priori* bound plays a crucial role in proving the main theorem.

**Lemma 2.6.** *Let  $\alpha_0 < \alpha < 1$  and  $1-\alpha < \beta < \min \left\{ \frac{\alpha}{2}, \frac{3\alpha-2}{2\alpha^2-6\alpha+5}, \frac{2-2\alpha}{4\alpha-3} \right\}$ . Assume that  $(u_0, \theta_0)$  satisfies the assumptions stated in Theorem 1.1, then the combined equation (2.4) admits the following bound for any  $0 \leq t \leq T$*

$$\|G(t)\|_{L^m}^m + \int_0^T \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m d\tau \leq C(T, u_0, \theta_0), \quad (2.18)$$

where  $m = \frac{2}{2\alpha-1} + \epsilon$  for some  $\epsilon > 0$  small enough, which may depend on  $\alpha$  and  $\beta$ .

**Remark 2.7.** It follows from the recent paper [42] (also [32]) that we need the key requirement  $m > \frac{2}{2\alpha-1}$ , but the  $m$  can be arbitrarily close to  $\frac{2}{2\alpha-1}$ . Thus it is sufficient to select  $m = \frac{2}{2\alpha-1} + \epsilon$  with any  $\epsilon > 0$  small enough.

**Proof of Lemma 2.6.** To begin with, let us recall the the following fractional version of the Gagliardo-Nirenberg inequality which is due to Hajaiej-Molinet-Ozawa-Wang [18]

$$\|\Lambda^{\gamma\beta} \theta\|_{L^{\frac{1}{\gamma}}} \leq C \|\Lambda^{\frac{\beta}{2}} \theta\|_{L^2}^{2\gamma} \|\theta\|_{L^\infty}^{1-2\gamma}, \quad 0 < \gamma < \frac{1}{2}.$$



In fact, the above inequality is a direct consequence of Theorem 1.2 of [18] as well as the equivalence  $\dot{W}^{s,p} \approx \dot{B}_{p,p}^s$  for  $0 < s \neq \mathbb{N}$  and  $1 < p < \infty$ .

Thanks to the bound (2.9), we have for any  $0 < \gamma < \frac{1}{2}$  and  $2 \leq q < \infty$

$$\begin{aligned}
\int_0^T \|\Lambda^{\gamma\beta}\theta(t)\|_{L^{\frac{1}{\gamma}}}^q dt &\leq C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q} \int_0^T \|\Lambda^{\frac{\beta}{2}}\theta(t)\|_{L^2}^{2\gamma q} dt \\
&\leq C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q} \int_0^T \|\Lambda^\delta\theta(t)\|_{L^2}^{\frac{4\delta\gamma q}{\beta}} \|\Lambda^{\delta+\frac{\beta}{2}}\theta(t)\|_{L^2}^{\frac{2(\beta-2\delta)\gamma q}{\beta}} dt \\
&\leq C\|\theta_0\|_{L^\infty}^{(1-2\gamma)q} \sup_{0 \leq t \leq T} \|\Lambda^\delta\theta(t)\|_{L^2}^{\frac{4\delta\gamma q}{\beta}} \int_0^T \|\Lambda^{\delta+\frac{\beta}{2}}\theta(t)\|_{L^2}^{\frac{2(\beta-2\delta)\gamma q}{\beta}} dt \\
&\leq C(T, u_0, \theta_0),
\end{aligned} \tag{2.19}$$

where in the last line we just take  $\delta$  such that  $\min\{\frac{\beta}{2}(1 - \frac{1}{q\gamma}), 0\} \leq \delta < \frac{\beta}{2}$ .

Multiplying the equation (2.4) by  $|G|^{m-2}G$  ( $m = \frac{2}{2\alpha-1} + \epsilon$  and  $\epsilon > 0$  to be fixed later), we have after integration by part and using the divergence-free condition

$$\begin{aligned}
&\frac{1}{m} \frac{d}{dt} \|G(t)\|_{L^m}^m + \int_{\mathbb{R}^2} (\Lambda^\alpha G) |G|^{m-2} G dx \\
&= \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u \cdot \nabla] \theta |G|^{m-2} G dx + \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \partial_x \theta |G|^{m-2} G dx \\
&= \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta |G|^{m-2} G dx + \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta |G|^{m-2} G dx \\
&\quad + \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \partial_x \theta |G|^{m-2} G dx.
\end{aligned} \tag{2.20}$$

We infer from the maximum principle and Sobolev embedding that

$$\int_{\mathbb{R}^2} (\Lambda^\alpha G) |G|^{m-2} G dx \geq \tilde{C} \|\Lambda^{\frac{\alpha}{2}} G^{\frac{m}{2}}\|_{L^2}^2 \geq \tilde{C} \|G\|_{L^{\frac{2m}{2-\alpha}}}^m, \tag{2.21}$$

where  $\tilde{C} > 0$  is an absolute constant.

Taking into account the inequality (2.8), we find that

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \Lambda^{\beta-\alpha} \partial_x \theta |G|^{m-2} G dx \right| &\leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|\Lambda^{1-\alpha+(1-\gamma)\beta}(|G|^{m-2}G)\|_{L^{\frac{1}{1-\gamma}}} \\
&\leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{B_{2, \frac{1}{1-\gamma}}^{1-\alpha+(1-\gamma)\beta}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2} \\
&\leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2},
\end{aligned} \tag{2.22}$$

where we have used  $H^{\frac{\alpha}{2}} \hookrightarrow B_{2, \frac{1}{1-\gamma}}^{1-\alpha+(1-\gamma)\beta}$  and  $1 - \alpha + (1 - \gamma)\beta < \frac{\alpha}{2}$ , namely

$$\gamma > \frac{2\beta + 2 - 3\alpha}{2\beta}. \tag{2.23}$$

Now the estimate (2.7) with  $s_1 = \gamma\beta$  implies that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta |G|^{m-2} G dx \right| \\
&\leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|\theta\|_{L^\infty} \| |G|^{m-2} G \|_{W^{s_2, \frac{1}{1-\gamma}}}
\end{aligned}$$

$$\begin{aligned}
& (s_2 > 2 - 2\alpha - \gamma\beta, \quad 0 \leq \gamma\beta < 1 - \alpha) \\
& \leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|\theta_0\|_{L^\infty} \|G\|_{B_{2, \frac{1}{1-\gamma}}^{s_2}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2} \\
& \leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2}, \quad (s_2 < \frac{\alpha}{2}).
\end{aligned} \tag{2.24}$$

Now we verify that the number of above  $s_2$  can be achieved. Indeed, it sufficient to select  $\gamma$  as follows

$$0 \leq \gamma\beta < 1 - \alpha, \quad 2 - 2\alpha - \gamma\beta < \frac{\alpha}{2}. \tag{2.25}$$

According to inequality (2.6) with  $s_1 = 0$  as well as inequality (2.8), it gives

$$\begin{aligned}
& \left| \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta |G|^{m-2} G dx \right| \\
& \leq C \|G\|_{L^q} \|\theta\|_{L^\infty} \| |G|^{m-2} G \|_{W^{s_2 - \frac{\tilde{\delta}}{2}, p}} \quad (s_2 - \frac{\tilde{\delta}}{2} > 1 - \alpha, \quad \frac{1}{p} + \frac{1}{q} = 1) \\
& \quad (\tilde{\delta} > 0 \text{ is small enough}) \\
& \leq C \|\theta_0\|_{L^\infty} \|G\|_{L^q} \|G\|_{L^{(m-2) \times \frac{q}{m-2}}}^{m-2} \|G\|_{B_{\frac{q}{q-(m-1)}, p}^{s_2 - \frac{\tilde{\delta}}{2}}} \quad (q > m-1) \\
& \leq C \|\theta_0\|_{L^\infty} \|G\|_{L^q}^{m-1} \|G\|_{B_{\frac{q}{q-(m-1)}, p}^{s_2 - \frac{\tilde{\delta}}{2}}} \\
& \leq C \|\theta_0\|_{L^\infty} \|G\|_{L^q}^{m-1} \|G\|_{H^{s_2-1+\frac{2(m-1)}{q}}} \quad (q \leq 2(m-1)),
\end{aligned} \tag{2.26}$$

where we have applied  $H^{s_2-1+\frac{2(m-1)}{q}} \hookrightarrow B_{\frac{q}{q-(m-1)}, p}^{s_2 - \frac{\tilde{\delta}}{2}}$  for  $m-1 < q \leq 2(m-1)$ . Thanks to the requirement  $s_2 - \frac{\tilde{\delta}}{2} > 1 - \alpha$  in (2.26), we can choose a sufficiently small  $\tilde{\delta} > 0$  (in fact we can take  $\tilde{\delta} \leq \frac{4\alpha-3}{8}$  for example to satisfy all the conditions) such that

$$s_2 = 1 - \alpha + \tilde{\delta}.$$

Notice that the following interpolation inequality

$$\|G\|_{H^{-\alpha+\tilde{\delta}+\frac{2(m-1)}{q}}} \leq C \|G\|_{L^2}^{1-\mu} \|G\|_{H^{\frac{\alpha}{2}}}^\mu,$$

where

$$\mu = \frac{-2\alpha + 2\tilde{\delta} + \frac{4(m-1)}{q}}{\alpha}, \quad \frac{4(m-1)}{3\alpha - 2\tilde{\delta}} \leq q \leq \frac{2(m-1)}{\alpha - \tilde{\delta}},$$

one can conclude that

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta |G|^{m-2} G dx \right| & \leq C \|\theta_0\|_{L^\infty} \|G\|_{L^q}^{m-1} \|G\|_{L^2}^{1-\mu} \|G\|_{H^{\frac{\alpha}{2}}}^\mu \\
& \leq C \|G\|_{L^q}^{m-1} \|G\|_{H^{\frac{\alpha}{2}}}^\mu.
\end{aligned} \tag{2.27}$$

Substituting the estimates (2.21)-(2.24) and (2.27) into (2.20), one arrives at

$$\frac{d}{dt} \|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m \leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2} + C \|G\|_{L^q}^{m-1} \|G\|_{H^{\frac{\alpha}{2}}}^\mu. \tag{2.28}$$

By the Gagliardo-Nirenberg inequalities, we know

$$\|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}} \leq C \|G\|_{L^m}^{1-\lambda_1} \|G\|_{L^{\frac{2m}{2-\alpha}}}^{\lambda_1}, \quad \lambda_1 = \frac{(1+2\gamma)m-4}{\alpha(m-2)}, \quad (2.29)$$

$$\|G\|_{L^q} \leq C \|G\|_{L^m}^{1-\lambda_2} \|G\|_{L^{\frac{2m}{2-\alpha}}}^{\lambda_2}, \quad \lambda_2 = \frac{2-\frac{2m}{q}}{\alpha}. \quad (2.30)$$

Here we want to emphasize that the following restrictions

$$\frac{4-m}{2m} \leq \gamma \leq \frac{m-(2-\alpha)(m-2)}{2m}, \quad m \leq q \leq \frac{2m}{2-\alpha} \quad (2.31)$$

implies  $0 \leq \lambda_1 \leq 1$  and  $0 \leq \lambda_2 \leq 1$ , respectively.

In view of above interpolation inequalities (2.29) and (2.30), we can obtain

$$\begin{aligned} & C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}} \|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2} \\ & \leq C \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}} \|G\|_{L^m}^{(m-2)(1-\lambda_1)} \|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-2)\lambda_1} \\ & \leq \frac{1}{4} \|G\|_{L^{\frac{2m}{2-\alpha}}}^m + C (\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}} \|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}}, \end{aligned} \quad (2.32)$$

$$\begin{aligned} & C \|G\|_{L^q}^{m-1} \|G\|_{H^{\frac{\alpha}{2}}}^\mu \\ & \leq C \|G\|_{L^m}^{(m-1)(1-\lambda_2)} \|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-1)\lambda_2} \|G\|_{H^{\frac{\alpha}{2}}}^\mu \\ & \leq \frac{1}{4} \|G\|_{L^{\frac{2m}{2-\alpha}}}^m + C \|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}} \|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned} \quad (2.33)$$

Inserting the estimates (2.32) and (2.33) into (2.28), it holds that

$$\begin{aligned} \frac{d}{dt} \|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m & \leq C (\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}} \|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}} \\ & \quad + C \|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}} \|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned}$$

By direct calculation, we have the following facts

$$\begin{aligned} \frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1} & \leq m, \quad \frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2} \leq m, \\ m & \leq \frac{2}{2-2\alpha+\delta} \Rightarrow \frac{m\mu}{m-(m-1)\lambda_2} \leq 2, \end{aligned}$$

and

$$\gamma < \frac{8-(2-\alpha)m}{4m} \quad \left(m < \frac{8}{2-\alpha}\right) \Rightarrow \frac{m}{m-(m-2)\lambda_1} < 2. \quad (2.34)$$

We thus get

$$\frac{d}{dt} \|G(t)\|_{L^m}^m \leq C \left\{ (\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}} + \|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}} \right\} (1 + \|G\|_{L^m}^m). \quad (2.35)$$

Thanks to the above facts (2.34) as well as the bound (2.19), we can deduce that

$$(\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}} \|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}} \in L^1(0, T), \quad \|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}} \in L^1(0, T).$$

By the Gronwall inequality, we can deduce from (2.35) that

$$\|G(t)\|_{L^m}^m + \int_0^T \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m d\tau \leq C < \infty. \quad (2.36)$$

Finally, let us check that all the restrictions would work. Combining all the requirement on the number  $q$ , it should be

$$\max \left\{ m-1, \frac{4(m-1)}{3\alpha - 2\tilde{\delta}}, m \right\} < q < \min \left\{ 2(m-1), \frac{2(m-1)}{\alpha - \tilde{\delta}}, \frac{2m}{2-\alpha} \right\}.$$

Direct computations yields that the number  $q$  can be fixed if we select  $\tilde{\delta} < \frac{3\alpha-2}{2}$ . Putting all the restrictions (2.23), (2.25), (2.31), (2.34) and  $0 < \gamma < \frac{1}{2}$  on  $\gamma$ , we have

$$\underline{\mathcal{B}}(\alpha) < \gamma < \overline{\mathcal{B}}(\alpha), \quad (2.37)$$

where

$$\begin{aligned} \underline{\mathcal{B}}(\alpha) &= \max \left\{ 0, \frac{2\beta + 2 - 3\alpha}{2\beta}, \frac{4 - 5\alpha}{2\beta}, \frac{4 - m}{2m} \right\}, \\ \overline{\mathcal{B}}(\alpha) &= \min \left\{ \frac{1}{2}, \frac{1 - \alpha}{\beta}, \frac{m - (2 - \alpha)(m - 2)}{2m}, \frac{8 - (2 - \alpha)m}{4m} \right\}, \end{aligned}$$

and

$$2 < m < \min \left\{ 4, \frac{2}{2 - 2\alpha + \tilde{\delta}}, \frac{8}{2 - \alpha} \right\} = 4.$$

According to  $\beta > 1 - \alpha$  and  $m < 4$ , the  $\underline{\mathcal{B}}(\alpha)$  and  $\overline{\mathcal{B}}(\alpha)$  can be reduced to

$$\begin{aligned} \underline{\mathcal{B}}(\alpha) &= \max \left\{ 0, \frac{2\beta + 2 - 3\alpha}{2\beta}, \frac{4 - m}{2m} \right\}, \\ \overline{\mathcal{B}}(\alpha) &= \min \left\{ \frac{1}{2}, \frac{1 - \alpha}{\beta}, \frac{m - (2 - \alpha)(m - 2)}{2m} \right\}. \end{aligned}$$

Therefore, the  $\gamma$  would work if the restriction on  $\beta$  satisfies

$$1 - \alpha < \beta < \min \left\{ \frac{\alpha}{2}, \frac{(3\alpha - 2)m}{m + (2 - \alpha)(m - 2)}, \frac{2(1 - \alpha)m}{4 - m} \right\}. \quad (2.38)$$

Notice that the above inequality (2.38) is strict inequality and the following key requirement

$$m > \frac{2}{2\alpha - 1}, \quad (2.39)$$

we just verify that the above inequality (2.38) holds true when  $m = \frac{2}{2\alpha - 1}$ . In this case, substituting the number  $m = \frac{2}{2\alpha - 1}$  into (2.38), the inequality (2.38) reduces to

$$1 - \alpha < \beta < \min \left\{ \frac{\alpha}{2}, \frac{3\alpha - 2}{2\alpha^2 - 6\alpha + 5}, \frac{2 - 2\alpha}{4\alpha - 3} \right\}. \quad (2.40)$$

By tedious computations, it is not difficult to check that  $\beta$  would work as long as

$$1 - \alpha < \frac{3\alpha - 2}{2\alpha^2 - 6\alpha + 5} \Rightarrow \alpha > \alpha_0.$$

If the above inequality (2.38) holds true when  $m = \frac{2}{2\alpha - 1}$ , then one may take  $m = \frac{2}{2\alpha - 1} + \epsilon$  for some sufficiently small  $\epsilon$  ( $\epsilon > 0$  may depend on  $\alpha$  and  $\beta$ ) such that both inequalities

(2.38) and (2.39) fulfil. The reason is that both the inequalities (2.38) and (2.39) are strict.  $\square$

Now let us say some words to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** In Lemma 2.6, we have proved that

$$\sup_{0 \leq t \leq T} \|G(t)\|_{L^{\frac{2}{2\alpha-1}+\epsilon}} < \infty, \quad (2.41)$$

which is a key estimate in order to complete the proof of Theorem 1.1 (see for example [32, 42]). For the sake of convenience, we sketch it here. In fact, as detailed in Step 2 of [42], the above estimate (2.41) implies

$$\int_0^T \|\omega(\tau)\|_{L^{\frac{2}{2\alpha-1}+\epsilon}} d\tau < \infty,$$

which further gives rise to

$$\int_0^T \|G(\tau)\|_{B_{\infty,1}^0} d\tau < \infty.$$

Finally, by Lemma 3.3 of [42], we obtain

$$\int_0^T \|\omega(\tau)\|_{B_{\infty,1}^0} d\tau < \infty.$$

It follows from the Littlewood-Paley technique that

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau \leq C \int_0^T (\|u(\tau)\|_{L^2} + \|\omega(\tau)\|_{B_{\infty,1}^0}) d\tau < \infty.$$

The above estimate is sufficient for us to get the desired results of Theorem 1.1. The details can be found in [32, 42]. Thus we omit the details. Therefore, this concludes the proof of Theorem 1.1.  $\square$

## REFERENCES

- [1] D. Adhikari, C. Cao, H. Shang, J. Wu, X. Xu, Z. Ye, Global regularity results for the 2D Boussinesq equations with partial dissipation, submitted for publication (2014).
- [2] D. Adhikari, C. Cao, J. Wu, The 2D Boussinesq equations with vertical viscosity and vertical diffusivity, J. Differential Equations 249 (2010) 1078-1088.
- [3] D. Adhikari, C. Cao, J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, J. Differential Equations 251 (2011) 1637-1655.
- [4] D. Adhikari, C. Cao, J. Wu, X. Xu, Small global solutions to the damped two-dimensional Boussinesq equations, J. Differential Equations 256 (2014) 3594-3613.
- [5] J. Cannon, E. DiBenedetto, The initial value problem for the Boussinesq equation with data in  $L^p$ , Lecture Notes in Mathematics, Vol. 771. Springer, Berlin, (1980), 129-144.
- [6] C. Cao, J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, Arch. Rational Mech 208 (2013), 985-1004.
- [7] M. Caputo, Linear models of dissipation whose  $Q$  is almost frequency independent-II, Geophy. J. R. Astr. Soc. 13 (1967), 529-539.
- [8] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Adv. Math. 203 (2006), 497-513.
- [9] D. Chae, J. Wu, The 2D Boussinesq equations with logarithmically supercritical velocities, Adv. Math. 230 (2012), 1618-1645.

- [10] P. Clavin, Instabilities and nonlinear patterns of overdriven detonations in gases, in: H. Berestycki, Y. Pomeau (Eds.), *Nonlinear PDEs in Condensed Matter and Reactive Flows*, Kluwer, (2002), 49-97.
- [11] P. Constantin, A. Majda, and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, *Nonlinearity* 7 (1994), 1495-1533.
- [12] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, *Geom. Funct. Anal.* 22 (2012), 1289-1321.
- [13] X. Cui, C. Dou, Q. Jiu, Local well-posedness and blow up criterion for the inviscid Boussinesq system in Hölder spaces, *J. Partial Differential Equations* 25 (2012), 220-238.
- [14] R. Danchin, Remarks on the lifespan of the solutions to some models of incompressible fluid mechanics, *Proc. Amer. Math. Soc.* 141 (2013), 1979-1993.
- [15] R. Danchin, M. Paicu, Global existence results for the anisotropic Boussinesq system in dimension two, *Math. Models Methods Appl. Sci.* 21 (2011), 421-457.
- [16] J. Droniou, C. Imbert, Fractal first-order partial differential equations, *Arch. Ration. Mech. Anal.* 182 (2006), 299-331.
- [17] S. Fan, A new extracting formula and a new distinguishing means on the one variable cubic equation, *J. Hainan Teach. Coll* 2 (1989) 91-98.
- [18] H. Hajaiej, L. Molinet, T. Ozawa, B. Wang, Sufficient and necessary conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized Boson equations, in: T. Ozawa, M. Sugimoto (Eds.), *RIMS Kkyroku Bessatsu B26: Harmonic Analysis and Nonlinear Partial Differential Equations*, Vol. 5, 2011, pp. 159-175.
- [19] Z. Hassainia, T. Hmidi, On the inviscid Boussinesq system with rough initial data, *J. Math. Anal. Appl.* 430 (2015), 777-809.
- [20] T. Hmidi, S. Keraani, F. Rousset, Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation, *J. Differential Equations* 249 (2010), 2147-2174.
- [21] T. Hmidi, S. Keraani, F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, *Comm. Partial Differential Equations* 36 (2011), 420-445.
- [22] T. Hmidi, On a maximum principle and its application to the logarithmically critical Boussinesq system, *Anal. PDE* (2011), 247-284.
- [23] T. Y. Hou, C. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Syst.* 12 (2005), 1-12.
- [24] J. Jia, J. Peng, K. Li, On the global well-posedness of a generalized 2D Boussinesq equations, *NoDEA Nonlinear Differential Equations Appl.* (in press).
- [25] Q. Jiu, C. Miao, J. Wu, Z. Zhang, The 2D incompressible Boussinesq equations with general critical dissipation, *SIAM J. Math. Anal.* 46 (2014) 3426-3454.
- [26] Q. Jiu, J. Wu, W. Yang, Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation, *J. Nonlinear Science* 25 (2015), 37-58.
- [27] D. KC, D. Regmi, L. Tao, J. Wu, The 2D Euler-Boussinesq equations with a singular velocity, *J. Differential Equations* 257 (2014) 82-108.
- [28] A. Larios, E. Lunasin, E.S. Titi, Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion, *J. Differential Equations* 255 (2013), 2636-2654.
- [29] J. Li, E.S. Titi, Global well-posedness of the 2D Boussinesq equations with vertical dissipation, [arXiv:1502.06180](https://arxiv.org/abs/1502.06180).
- [30] X. Liu, M. Wang, Z. Zhang, Local well-posedness and blowup criterion of the Boussinesq equations in critical Besov spaces, *J. Math. Fluid Mech.* 12 (2010) 280-292.
- [31] A. Majda, A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2001.
- [32] C. Miao, L. Xue, On the global well-posedness of a class of Boussinesq-Navier-Stokes systems, *NoDEA Nonlinear Differential Equations Appl.* 18 (2011), 707-735.
- [33] J. Pedlosky, *Geophysical fluid dynamics*, New York, Springer-Verlag, 1987.
- [34] A. Stefanov, J. Wu, A global regularity result for the 2D Boussinesq equations with critical dissipation, [arXiv:1411.1362v3 \[math.AP\]](https://arxiv.org/abs/1411.1362v3).
- [35] J. Wu, X. Xu, Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation, *Nonlinearity* (2014) 2215-2232.

- [36] J. Wu, X. Xu, Z. Ye, Global smooth solutions to the  $n$ -dimensional damped models of incompressible fluid mechanics with small initial datum, *J. Nonlinear Science* 25 (2015), 157-192.
- [37] X. Xu, Global regularity of solutions of 2D Boussinesq equations with fractional diffusion, *Nonlinear Anal.* 72 (2010), 677-681.
- [38] W. Yang, Q. Jiu, J. Wu, Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation, *J. Differential Equations*, 257 (2014) 4188-4213.
- [39] Z. Ye, Blow-up criterion of smooth solutions for the Boussinesq equations, *Nonlinear Anal.* 110 (2014), 97-103.
- [40] Z. Ye, X. Xu, Remarks on global regularity of the 2D Boussinesq equations with fractional dissipation, *Nonlinear Anal.* (in press).
- [41] Z. Ye, X. Xu, Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation, arXiv:1506.00470v1 [math.AP].
- [42] Z. Ye, X. Xu, L. Xue, On the global regularity of the 2D Boussinesq equations with fractional dissipation, submitted for publication (2014).